Received: *May 30, 2000*, Accepted: *July 24, 2000*

Perturbative Chern-Simons theory on non-commutative $\mathbb{R}^3$

Andreas A. Bichl, Jesper M. Grimstrup,* Volkmar Putz and Manfred Schweda†

Institut für Theoretische Physik, Technische Universität Wien
Wiedner Hauptstraße 8-10, A-1040 Wien, Austria
E-mail: bichl@hep.itp.tuwien.ac.at, jesper@hep.itp.tuwien.ac.at, e9626026@student.tuwien.ac.a, mschweda@email.tuwien.ac.at

Abstract: A U(N) Chern-Simons theory on non-commutative $\mathbb{R}^3$ is constructed as a θ-deformed field theory. The model is characterized by two symmetries: the BRST-symmetry and the topological linear vector supersymmetry. It is shown that the theory is finite and θ-independent at the one-loop level and that the calculations respect the restriction of the topological supersymmetry. Thus the topological θ-deformed Chern-Simons theory is an example of a model which is non singular in the limit $\theta \to 0$.

Keywords: Topological Field Theories, Chern-Simons Theories, Non-Commutative Geometry.

*Work supported by "The Danish Research Academy".

†Work supported by the "Fond zur Förderung der Wissenschaftlichen Forschung" under Project Grant Number P11582-PHY.

Contents

1. Introduction

It is well known that at the Planck scale the flat space-time must be modified drastically leading to a non-commutative space-time. The corresponding field theories have to be formulated in the framework of non-commutative geometry [1].

In this short letter we discuss the Chern-Simons theory with respect to a non-abelian $\mathrm{U}(N)$-gauge group on non-commutative $\mathbb{R}^3$. In the corresponding θ-deformed field theory we are able to establish the usual BRST-symmetry [2] and the topological vector supersymmetry (VSUSY) [3]. The vector-like generators of this VSUSY give rise together with the BRST-operator to a Wess-Zumino-like anticommutation relation which closes on-shell on the space-time translations.

At the classical level the Ward-identity characterizing the VSUSY is linearly broken in the quantum fields. These breakings are induced by the external sources needed for the description of the non-linear terms of the BRST-transformations.

It is well known that in the commutative case the perturbative finiteness of the Chern-Simons model is governed by the VSUSY within the algebraic renormalization procedure [3].

In the presented notes we are able to show that the Chern-Simons model is finite and independent of the deformation parameter θ at least at the one-loop level. Additionally, the perturbative calculations are in agreement with the restrictions coming from the VSUSY.

The letter is organized as follows. Section 2 gives a short presentation of θ-deformed field theory. In section 3 we present the Chern-Simons theory as a θ-deformed field theory on a non-commutative $\mathbb{R}^3$. The BRST-transformation and the VSUSY are introduced at the tree level. Section 4 is devoted to sketch the perturbative calculation at the one-loop level.

JHEP07(2000)046

2. θ-deformed field theory

The non-commutative $\mathbb{R}^3$ is defined as the algebra $\mathcal{A}_x$ generated by $\widehat{x}_\mu$, $\mu = 1, 2, 3$ satisfying the commutation relation [4]

$$[\widehat{x}_\mu, \widehat{x}_\nu] = i\theta_{\mu\nu}\,, \tag{2.1}$$

where $\theta_{\mu\nu}$ is a real, constant, antisymmetric matrix with rank 3. For a function $f(x)$ on ordinary $\mathbb{R}^3$ one associates an element $W(f)$ of the algebra $\mathcal{A}_x$ by

$$W(f) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, e^{ik_\mu \widehat{x}_\mu} \tilde{f}(k)\,, \tag{2.2}$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$

$$\tilde{f}(k) = \frac{1}{(2\pi)^{3/2}} \int d^3x\, e^{-ik_\mu x_\mu} f(x)\,. \tag{2.3}$$

Using relation (2.2) the star product on $\mathbb{R}^3$ is defined by

$$W(f)W(g) = W(f \star g)\,, \tag{2.4}$$

and one finds

$$(f \star g)(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i(k_\mu + p_\mu)x_\mu} e^{-\frac{i}{2}\theta_{\mu\nu}k_\mu p_\nu} \tilde{f}(k)\tilde{g}(p)\,. \tag{2.5}$$

Relations (2.2) and (2.4) render a representation of functions on the algebra $\mathcal{A}_x$ [5], and thus allow us to examine gauge theory on $\mathcal{A}_x$ by considering the counterpart on ordinary $\mathbb{R}^3$ with the usual product replaced by the star product. From (2.5) follows

$$\int d^3x (f \star g)(x) = \int d^3x f(x) g(x)\,. \tag{2.6}$$

Similarly, one gets for a triple product

$$\begin{aligned}(f \star g \star h)(x) = & \int \frac{d^3p_1}{(2\pi)^{3/2}} \frac{d^3p_2}{(2\pi)^{3/2}} \frac{d^3p_3}{(2\pi)^{3/2}}\, e^{-\frac{i}{2}\theta_{\mu\nu}[(p_1)_\mu(p_2)_\nu + (p_1)_\mu(p_3)_\nu + (p_2)_\mu(p_3)_\nu]} \times \\ & \times e^{i(p_1+p_2+p_3)_\mu x_\mu} \tilde{f}(p_1)\tilde{g}(p_2)\tilde{h}(p_3)\,.\end{aligned} \tag{2.7}$$

With n arbitrary fields ϕ_k one can show the validity of cyclic permutations

$$\begin{aligned}\int_{\mathbb{R}^3} d^3x (\phi_1 \star \phi_2 \star \cdots \star \phi_n)(x) = & (-1)^{g_1(g_2 + \cdots + g_n)} \times \\ & \times \int_{\mathbb{R}^3} d^3x (\phi_2 \star \cdots \star \phi_n \star \phi_1)(x)\,,\end{aligned} \tag{2.8}$$

where g_i is the total grading of the field ϕ_i. Additionally, one can show that the functional differentiation holds too, i.e.

$$\frac{\delta}{\delta\phi_1(y)} \int_{\mathbb{R}^3} d^3x (\phi_1 \star \phi_2 \star \cdots \star \phi_n)(x) = (\phi_2 \star \cdots \star \phi_n)(y)\,. \tag{2.9}$$

Equations (2.8) and (2.9) imply now that the off-shell algebra of the relevant functional operators describing the symmetry content of the model remains valid.

JHEP07(2000)046

3. BRST-symmetry and linear VSUSY

With a Lie-algebra valued gauge field $A_\mu(x) = A^a_\mu(x)T^a$, where T^a are the N^2 generators of the U(N)-group,[1] $a = 0, 1, \ldots, (N^2 - 1)$

$$\left(T^0\right)_{mn} = \left(\frac{2}{N}\right)^{1/2} \delta_{mn}\,, \qquad \left(T^a\right)_{mn}\,, \tag{3.1}$$

with

$$\left[T^a, T^b\right] = 2if^{abc}T^c\,, \qquad \left\{T^a, T^b\right\} = 2d^{abc}T^c\,, \qquad \mathrm{Tr}\left(T^aT^b\right) = 2\delta^{ab}\,, \tag{3.2}$$

one can define the gauge-invariant classical metric independent action as

$$\Sigma_{\mathrm{cl}} = -\int d^3x\, \frac{1}{2}\epsilon_{\mu\nu\rho}\, \mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu \star A_\nu \star A_\rho\right), \tag{3.3}$$

which is invariant under the infinitesimal gauge transformation

$$\delta A_\mu = \partial_\mu\lambda + i\left(\lambda \star A_\mu - A_\mu \star \lambda\right) \equiv D_\mu\lambda\,, \tag{3.4}$$

where λ is a Lie-algebra valued gauge parameter. The corresponding field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left(A_\mu \star A_\nu - A_\nu \star A_\mu\right). \tag{3.5}$$

In order to quantize the model within the BRST-scheme, the gauge-symmetry is replaced by the nilpotent BRST-symmetry [2, 3]

$$sA_\mu = D_\mu c\,, \qquad sc = ic \star c\,, \tag{3.6}$$

where c is the anticommuting Faddeev-Popov ghost field. Within the quantization procedure a BRST-invariant gauge-fixing must be introduced in the following manner:

$$\Sigma_{\mathrm{gf}} = s\int d^3x\, \mathrm{Tr}\left(\bar{c} \star \partial_\mu A_\mu\right), \tag{3.7}$$

with

$$s\bar{c} = B\,, \qquad sB = 0\,, \tag{3.8}$$

where $\bar{c}$ is the antighost field and B the multiplier field implementing the Landau gauge. The gauge fixing part of the action of course depends on the metric, chosen here as the flat euclidean one $\delta_{\mu\nu}$. The total action is now

$$\Sigma = \Sigma_{\mathrm{cl}} + \Sigma_{\mathrm{gf}}\,. \tag{3.9}$$

[1]Not all gauge groups are realizable in non-commutative field theory [5, 6, 7].

JHEP07(2000)046

Besides the BRST-invariance, (3.9) possesses an additional global supersymmetry, whose generators carry the Lorentz index [3]

$$\delta_\nu A_\tau = \epsilon_{\mu\nu\tau}\partial_\mu \bar{c}\,, \qquad \delta_\nu c = -A_\nu\,, \qquad \delta_\nu B = -\partial_\nu \bar{c}\,, \qquad \delta_\nu \bar{c} = 0\,. \tag{3.10}$$

The operator δ_ν gives rise together with the BRST-operator to the anticommutation relations

$$\begin{aligned} \delta_\nu \delta_\tau + \delta_\tau \delta_\nu &= 0\,, \\ \delta_\nu s + s\delta_\nu &= -\partial_\nu + (\text{eq. of motion})\,, \end{aligned} \tag{3.11}$$

which close on-shell on space time translations. In order to describe the symmetry content of the model with respect to the BRST-symmetry and the linear VSUSY, one has to write down the corresponding Ward-identities (WI) at the classical level. In order to carry out this procedure, one introduces external unquantized sources for the non-linear pieces of the BRST-transformations

$$\Sigma_{\text{ext}} = \int d^3x\, \mathrm{Tr}\left(\rho_\mu \star sA_\mu + \sigma \star sc\right). \tag{3.12}$$

Then the total corresponding tree-level action is given by

$$\begin{aligned} \Sigma &= \Sigma_{\text{cl}} + \Sigma_{\text{gf}} + \Sigma_{\text{ext}} \\ &= -\int d^3x\, \frac{1}{2}\epsilon_{\mu\nu\rho}\, \mathrm{Tr}\left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu \star A_\nu \star A_\rho\right) + \\ &\quad + \int d^3x\, \mathrm{Tr}\left(B\partial_\mu A_\mu - \bar{c}\star \partial_\mu D_\mu c\right) + \\ &\quad + \int d^3x\, \mathrm{Tr}\left(\rho_\mu \star D_\mu c + i\sigma \star c \star c\right). \end{aligned} \tag{3.13}$$

The total action satisfies the non-linear, θ-deformed Slavnov-Taylor identity

$$\mathcal{S}\left(\Sigma\right) = 0\,, \tag{3.14}$$

where

$$\mathcal{S}\left(\Sigma\right) = \int d^3x\, \mathrm{Tr}\left(\frac{\delta\Sigma}{\delta\rho_\mu} \star \frac{\delta\Sigma}{\delta A_\mu} + \frac{\delta\Sigma}{\delta\sigma} \star \frac{\delta\Sigma}{\delta c} + B \star \frac{\delta\Sigma}{\delta\bar{c}}\right), \tag{3.15}$$

which describes the BRST-symmetry. The VSUSY is characterized by the following integrated functional differential operator:

$$\mathcal{W}_\mu = \int d^3x\, \mathrm{Tr}\left(\epsilon_{\nu\mu\tau}\left(\partial_\nu \bar{c} + \rho_\nu\right) \star \frac{\delta}{\delta A_\tau} - A_\mu \star \frac{\delta}{\delta c} - \partial_\mu \bar{c} \star \frac{\delta}{\delta B} - \sigma \star \frac{\delta}{\delta \rho_\mu}\right). \tag{3.16}$$

The WI operator (3.16) yields the linearly broken supersymmetry

$$\mathcal{W}_\mu \Sigma = \Delta_\mu^{\text{cl}}\,, \tag{3.17}$$

with

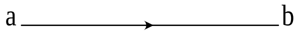
$$\Delta_\mu^{\text{cl}} = \int d^3x\, \mathrm{Tr}\left(-\epsilon_{\nu\mu\tau}\rho_\nu \partial_\tau B + \rho_\tau \partial_\mu A_\tau - \sigma\partial_\mu c\right), \tag{3.18}$$

where Δ_μ^{cl} is linear in the quantum fields.

JHEP07(2000)046

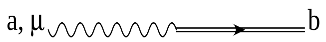
4. One-loop calculations

In order to check the one-loop UV and IR behaviour of the non-commutative Chern-Simons model, one needs the corresponding Feynman rules. For the various propagators of the model only the bilinear part of the full action (3.13) is relevant. However, due to the fact that noncommutativity does not affect bilinear parts of the action (2.6), the propagators are the usual ones [8]. In momentum space one therefore has



$$\tilde{G}^{c\bar{c},ab}(k) = \frac{\delta^{ab}}{(2\pi)^{3/2}}\frac{1}{k^2}\,, \tag{4.1}$$

$$\tilde{G}^{AA,ab}_{\mu\nu}(k) = \frac{i\delta^{ab}}{(2\pi)^{3/2}}\frac{\epsilon_{\mu\nu\lambda}k_\lambda}{k^2}\,, \tag{4.2}$$



$$\tilde{G}^{AB,ab}_{\mu}(k) = -\frac{i\delta^{ab}}{(2\pi)^{3/2}}\frac{k_\mu}{k^2}\,. \tag{4.3}$$

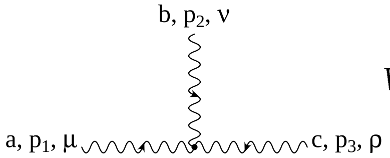
The interaction piece of the action (3.13) determines the corresponding Feynman-rules for the three gluon vertex and the ghost-antighost-gluon vertex:

$$\begin{aligned}\Sigma_{\text{int}} &= \int d^3x\,\mathrm{Tr}\left(\frac{i}{3}\epsilon_{\mu\nu\rho}A_\mu\star A_\nu\star A_\rho + i\partial_\mu\bar{c}\star(c\star A_\mu - A_\mu\star c)\right) \\ &= \mathrm{Tr}^{abc}\int d^3x\left(\frac{i}{3}\epsilon_{\mu\nu\rho}A^a_\mu\star A^b_\nu\star A^c_\rho + i\partial_\mu\bar{c}^a\star c^b\star A^c_\mu - i\partial_\mu\bar{c}^a\star A^b_\mu\star c^c\right),\end{aligned} \tag{4.4}$$

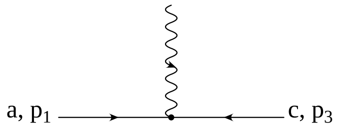
where $\mathrm{Tr}^{abc} = \mathrm{Tr}(T^aT^bT^c)$. From (3.2) follows

$$\mathrm{Tr}^{abc} = 2\left(if^{abc} + d^{abc}\right), \tag{4.5}$$

and therefore the corresponding Feynman rules for the interaction vertices in momentum space become



$$\tilde{V}^{AAA,abc}_{\mu\nu\rho}(p_1,p_2,p_3) = \frac{1}{(2\pi)^{3/2}}\epsilon_{\mu\nu\rho}\Omega^{abc}_\theta(p_2,p_3)\,, \tag{4.6}$$

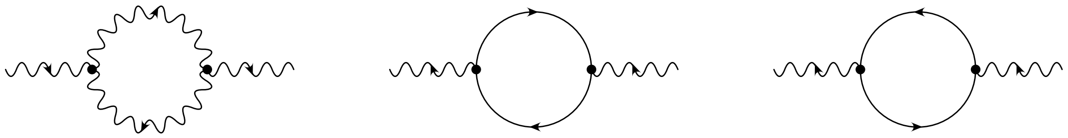


$$\tilde{V}^{cA\bar{c},abc}_{\mu}(p_1,p_2,p_3) = -\frac{1}{(2\pi)^{3/2}}i(p_3)_\mu\Omega^{abc}_\theta(p_2,p_3)\,, \tag{4.7}$$

where $\Omega^{abc}_\theta(p_2,p_3)$ is defined by

$$\Omega^{abc}_\theta(p_2,p_3) = 4\left[f^{abc}\cos\left(\frac{1}{2}p_2\times p_3\right) - d^{abc}\sin\left(\frac{1}{2}p_2\times p_3\right)\right], \tag{4.8}$$

JHEP07(2000)046



(a) (b) (c)

Figure 1: Gluon self energy.

with $p_2 \times p_3 = \theta_{\mu\nu}(p_2)_\mu (p_3)_\nu$. In order to carry out explicit calculations the following contractions of the structure constants f^{abc} and d^{abc} are useful

$$f^{abc} f^{dbc} = N\left(\delta^{ad} - \delta^{a0}\delta^{d0}\right), \tag{4.9}$$
$$d^{abc} d^{dbc} = N\left(\delta^{ad} + \delta^{a0}\delta^{d0}\right). \tag{4.10}$$

From the commutation relations we have additionally,

$$f^{abc} = \frac{1}{4i}\,\mathrm{Tr}\left(\left[T^a, T^b\right] T^c\right), \qquad d^{abc} = \frac{1}{4}\,\mathrm{Tr}\left(\left\{T^a, T^b\right\} T^c\right). \tag{4.11}$$

In order to verify the relations (4.9) and (4.10) one has to use the following completeness relations for the generators T^a of the $\mathrm{U}(N)$ gauge group

$$\left(T^a\right)_{mn}\left(T^a\right)_{m'n'} = 2\delta_{mn'}\delta_{nm'}\,, \quad m, n = 1, \ldots, N\,. \tag{4.12}$$

Before doing explicitly the one-loop analysis we want to stress that already at the tree level the consequences of the VSUSY WI are fulfilled. In doing a Legendre transformation one gets from the WI (3.17) that the propagator of the gluon field and the ghost-antighost-field propagator are related. In momentum space one obtains

$$\epsilon_{\mu\nu\lambda} i k_\lambda \tilde{G}^{c\bar{c},ab}(k) = \tilde{G}^{AA,ab}_{\mu\nu}(k)\,. \tag{4.13}$$

For the three point vertices the following WI follow directly from (3.17)

$$i(p_3)_\nu \epsilon_{\mu\nu\tau} \tilde{V}^{AAA,abc}_{\lambda\sigma\tau}(p_1,p_2,p_3) = \delta_{\mu\sigma}\tilde{V}^{cA\bar{c},abc}_{\lambda}(p_1,p_2,p_3) - \delta_{\mu\lambda}\tilde{V}^{cA\bar{c},abc}_{\sigma}(p_1,p_2,p_3)\,. \tag{4.14}$$

These relations are indeed responsible for the one-loop perturbative finiteness. In fact, one finds that the gluon loop and the ghost loop contribution to the gluon self energy $\Pi^{ab}_{\mu\nu}$ are equal up to a sign, and thus cancel. A similar result is found for the one-loop vertex corrections which therefore also vanish. Therefore, the loop cancellation present in the ordinary Chern-Simons model [8] persists in the non-commutative case, i.e. the non-commutative Chern-Simons model is finite to first loop order.

In the following we will explicitely perform the one-loop analysis of the gluon self energy graph (a) of figure 1 and show that the "non-planar" contribution arising due to the noncommutativity is finite. Using the above Feynman rules one calculates

JHEP07(2000)046

the following expression for the gluon self-energy with an internal gluon field,

$$\Pi^{ab}_{\mu\nu}(p,-p) = -\frac{1}{(2\pi)^6}\int \frac{d^3k}{(2\pi)^{3/2}} \frac{(k_+)_\mu(k_-)_\nu + (k_+)_\nu(k_-)_\mu}{k_+^2 k_-^2} \times \\ \times \Omega_\theta^{acd}(-k_+,k_-)\Omega_\theta^{cbd}(-p,-k_-)\,, \tag{4.15}$$

where $k_+ = k + p/2$ and $k_- = k - p/2$. Using now (4.9) and (4.10) one gets

$$\Pi^{ab}_{\mu\nu}(p,-p) = \frac{16N}{(2\pi)^6}\int \frac{d^3k}{(2\pi)^{3/2}} \frac{(k_+)_\mu(k_-)_\nu + (k_+)_\nu(k_-)_\mu}{k_+^2 k_-^2} \times \\ \times \left[\delta^{ab} - \delta^{a0}\delta^{b0}\frac{1}{2}\left(e^{ip\times k} + e^{-ip\times k}\right)\right]. \tag{4.16}$$

The first "planar" term is just the result which corresponds to the commutative model. From the result (4.16) it is also seen that one has to use a $\mathrm{U}(N)$ gauge group in order to have a non-trivial result. Now one can use the result of [8] with an appropriate regularization scheme in order to discuss the θ-independent expression of eq. (4.16). The θ-independent terms, the so-called planar contributions, represent the usual effects of the commutative model. These individual one-loop diagrams develop at most linear (actually logarithmic) divergences. In order to avoid difficulties with the ϵ-tensor in connection with the dimensional regularization and since all relevant propagators (see eqs. (4.1), (4.2) and (4.3)) depend on $1/k^2$ one may simply use the Pauli-Villars regularization scheme in order to regularize the planar one-loop graphs [8].

For the evaluation of the non-planar contributions no regularization procedure is needed since these graphs are perturbatively finite as it is seen from the explicit calculations. Using Schwinger parametrization

$$\frac{1}{k_+^2} = \int_0^\infty d\alpha_+ e^{-\alpha_+ k_+^2} \tag{4.17}$$

we discuss now

$$\Pi'_{\mu\nu}(p,-p) = -\int \frac{d^3k}{(2\pi)^{3/2}} \frac{(k_+)_\mu(k_-)_\nu + (k_+)_\nu(k_-)_\mu}{k_+^2 k_-^2}\left(e^{ip\times k} + e^{-ip\times k}\right), \tag{4.18}$$

and get

$$\Pi'_{\mu\nu}(p,-p) = -\int \frac{d^3k}{(2\pi)^{3/2}}\int_0^\infty d\alpha_+ \int_0^\infty d\alpha_- \Big((k_+)_\mu(k_-)_\nu + (k_+)_\nu(k_-)_\mu\Big)e^{-\alpha_+k_+^2-\alpha_-k_-^2} \times \\ \times \left(e^{ip\times k} + e^{-ip\times k}\right). \tag{4.19}$$

In defining $\tilde{p}_\mu = p_\lambda\theta_{\lambda\mu}$ and

$$I_{\mu\nu}(\eta,p) = \left\{\left(\frac{\partial}{\partial y_\mu}\frac{\partial}{\partial z_\nu} + \frac{\partial}{\partial y_\nu}\frac{\partial}{\partial z_\mu}\right) \times \\ \times \int \frac{d^3k}{(2\pi)^{3/2}} e^{-\alpha_+(k+\frac{p}{2})^2 - \alpha_-(k-\frac{p}{2})^2 + i\eta\tilde{p}k - y(k+\frac{p}{2}) - z(k-\frac{p}{2})}\right\}\Bigg|_{y=z=0}. \tag{4.20}$$

JHEP07(2000)046

Equation (4.19) can be written as

$$\Pi'_{\mu\nu}(p,-p) = -\int_0^\infty d\alpha_+ \int_0^\infty d\alpha_- \Big(I_{\mu\nu}(1,p) + I_{\mu\nu}(-1,p) \Big) . \tag{4.21}$$

In order to carry out the k-integration one introduces new variables

$$k'_\lambda = k_\lambda + \frac{B_\lambda}{\alpha_+ + \alpha_-} \equiv k_\lambda + \frac{B_\lambda}{\beta} , \tag{4.22}$$

with

$$B_\lambda = \frac{1}{2}\Big[(\alpha_+ - \alpha_-)\, p_\lambda - i\eta\tilde{p}_\lambda + (y+z)_\lambda \Big] , \tag{4.23}$$

in order to use gaussian integration

$$\int \frac{d^3k'}{(2\pi)^{3/2}} e^{-\beta k'^2} = \frac{1}{(2\beta)^{3/2}} . \tag{4.24}$$

In this way one gets for (4.20)

$$I_{\mu\nu}(\eta,p) = \left\{ \left(\frac{\partial}{\partial y_\mu}\frac{\partial}{\partial z_\nu} + \frac{\partial}{\partial y_\nu}\frac{\partial}{\partial z_\mu} \right) \frac{1}{(2\beta)^{3/2}} e^{\frac{B^2}{\beta} - \frac{\beta p^2}{4} - (y-z)\frac{p}{2}} \right\}\Bigg|_{y=z=0} . \tag{4.25}$$

Doing explicitly the differentiation one finds

$$\begin{aligned} I_{\mu\nu}(\eta,p) = \frac{1}{(2\beta)^{3/2}} e^{-\frac{\alpha_+\alpha_-}{\beta}p^2 - \frac{\eta^2}{4\beta}\tilde{p}^2} \Bigg\{ & \frac{1}{\beta}\delta_{\mu\nu} - 2\left(\frac{\alpha_+\alpha_-}{\beta^2} p_\mu p_\nu + \eta^2 \frac{\tilde{p}_\mu\tilde{p}_\nu}{4\beta^2} \right) + \\ & + \text{terms linear in } \eta \Bigg\} , \end{aligned} \tag{4.26}$$

implying the following result:

$$\begin{aligned} \Pi'_{\mu\nu}(p,-p) = -2\int_0^\infty d\alpha_+ \int_0^\infty d\alpha_- e^{-\frac{\alpha_+\alpha_-}{\beta}p^2 - \frac{\tilde{p}^2}{4\beta}} \frac{1}{(2\beta)^{3/2}} \times \\ \times \left\{ \frac{\delta_{\mu\nu}}{\beta} - 2\left(\frac{\alpha_+\alpha_-}{\beta^2} p_\mu p_\nu + \frac{\tilde{p}_\mu\tilde{p}_\nu}{2\beta^2} \right) \right\} . \end{aligned} \tag{4.27}$$

It remains to study the parametric integration (4.27). One has two different types of integrals (with $\alpha = 1, 2$)

$$I_\alpha^{(i)} = \int_0^\infty d\alpha_+ \int_0^\infty d\alpha_- \frac{1}{\beta^{\frac{3}{2}+\alpha}} e^{-\frac{\alpha_+\alpha_-}{\beta}p^2 - \frac{\tilde{p}^2}{4\beta}} , \tag{4.28}$$

$$I^{(ii)} = \int_0^\infty d\alpha_+ \int_0^\infty d\alpha_- \frac{\alpha_+\alpha_-}{\beta^{7/2}} e^{-\frac{\alpha_+\alpha_-}{\beta}p^2 - \frac{\tilde{p}^2}{4\beta}} . \tag{4.29}$$

With the reparametrization

$$\alpha_+ = \xi\lambda , \qquad \alpha_- = (1-\xi)\lambda , \qquad \rho = \xi(1-\xi)p^2\lambda , \tag{4.30}$$

JHEP07(2000)046

one arrives at

$$I_\alpha^{(i)} = \int_0^1 d\xi \left(\xi(1-\xi)p^2\right)^{(\alpha-1/2)} \int_0^\infty d\rho \frac{1}{\rho^{(1/2+\alpha)}} e^{-(\rho+\frac{\xi(1-\xi)p^2\tilde{p}^2}{4\rho})} \tag{4.31}$$

and

$$I^{(ii)} = \left(\frac{1}{p^2}\right)^{1/2} \int_0^1 d\xi \, [\xi(1-\xi)]^{1/2} \int_0^\infty \frac{d\rho}{\rho^{1/2}} e^{-(\rho+\frac{\xi(1-\xi)p^2\tilde{p}^2}{4\rho})} . \tag{4.32}$$

The ρ-integration is carried out with the formula [12]

$$\int_0^\infty \frac{dx}{x^{1-\nu}} e^{-\gamma x - \beta/x} = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu\left(2\sqrt{\beta\gamma}\right), \tag{4.33}$$

where K_ν is the modified Bessel function [12, 13]. In this way one obtains

$$I_\alpha^{(i)} = 2\left(\frac{4p^2}{\tilde{p}^2}\right)^{(\alpha-1/2)\frac{1}{2}} \int_0^1 d\xi \, [\xi(1-\xi)]^{(\alpha-1/2)\frac{1}{2}} K_{1/2-\alpha}(z) , \tag{4.34}$$

where $z = (\xi(1-\xi)p^2\tilde{p}^2)^{1/2}$. Additionally, one has

$$I^{(ii)} = \left(\frac{1}{p^2}\right)^{1/2} \sqrt{2} \int_0^1 d\xi \, [\xi(1-\xi)]^{1/2} \left[\xi(1-\xi)p^2\tilde{p}^2\right]^{1/4} K_{1/2}(z) . \tag{4.35}$$

With

$$K_{1/2}(z) = K_{-1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \tag{4.36}$$

and

$$K_{\nu-1}(z) - K_{\nu+1}(z) = \frac{-2\nu}{z} K_\nu(z) , \tag{4.37}$$

one can show that $I_1^{(i)}$ behaves as

$$I_1^{(i)} \sim \left(\frac{1}{\tilde{p}^2}\right)^{1/2} \times \text{finite} , \tag{4.38}$$

which shows the usual IR-singularity for $\tilde{p} \to 0$. Whereas $I_2^{(i)}$ develops two contributions of the form

$$I_2^{(i)} \sim \left(p^2\right)^{1/2} \left(\frac{1}{\tilde{p}^2}\right) \times \text{finite} + \left(\frac{1}{\tilde{p}^2}\right)^{3/2} \times \text{finite} . \tag{4.39}$$

These last terms must be multiplied by $\tilde{p}_\mu \tilde{p}_\nu$. Thus the second term of (4.39) reproduces an IR-singularity for $\tilde{p} \to 0$, whereas the first term leads to IR-regular, θ-independent matrix elements.

A similar calculation for $I^{(ii)}$ yields

$$I^{(ii)} \sim \left(\frac{1}{p^2}\right)^{1/2} \times \text{finite} , \tag{4.40}$$

a θ-independent result. By setting $\theta = 0$ the second term of (4.16) will produce a term proportional to $\delta^{0a}\delta^{0b}$, (4.40). This term combines with the first part of (4.16) to reproduce the result of the commutative SU(N) model.

JHEP07(2000)046

5. Conclusion and outlook

In this short note we have demonstrated the finiteness of the θ-deformed Chern-Simons theory and its θ-independence at the one-loop level.

Additionally, the obtained results are in agreement with the restrictions coming from the VSUSY. In the limit $\theta \to 0$ one gets the old results of [8]. Furthermore, since no radiative corrections are present, the model is regular in this limit (see also [10, 11]).

In a forthcoming paper we will discuss further topological field models of Schwarz-type and Witten-type.

Acknowledgments

It is our pleasure to thank H. Grosse, H. Hüffel, C. Rupp and R. Wulkenhaar for helpful discussions in a weekly seminar.

References

[1] S. Doplicher, K. Fredenhagen and J.E. Roberts, *Space-time quantization induced by classical gravity*, *Phys. Lett.* **B 331** (1994) 39; *The quantum structure of space-time at the Planck scale and quantum fields*, *Comm. Math. Phys.* **172** (1995) 187.

[2] C. Becchi, A. Rouet and R. Stora, *The abelian Higgs-Kibble model. Unitarity of the s operator*, *Phys. Lett.* **B 52** (1974) 344; *Renormalization of the abelian Higgs-Kibble model*, *Comm. Math. Phys.* **42** (1975) 127; *Renormalization of gauge theories*, *Ann. Phys. (NY)* **98** (1976) 287.

[3] O. Piguet and S.P. Sorella, *Algebraic renormalization: perturbative renormalization, symmetries and anomalies*, Springer, Berlin 1995;
A. Boresch, S. Emery, O. Moritsch, M. Schweda, T. Sommer and H. Zerrouki, *Applications of noncovariant gauges in the algebraic renormalization procedure*, World Scientific, London 1998.

[4] T. Filk, *Divergencies in a field theory on quantum space*, *Phys. Lett.* **B 376** (1996) 53.

[5] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*, *Eur. Phys. J.* **C 16** (2000) 161 [hep-th/0001203].

[6] K. Matsubara, *Restrictions on gauge groups in noncommutative gauge theory*, *Phys. Lett.* **B 482** (2000) 417 [hep-th/0003294].

[7] S. Terashima, *A note on superfields and noncommutative geometry*, *Phys. Lett.* **B 482** (2000) 276 [hep-th/0002119].

[8] E. Guadagnini, M. Martellini and M. Mintchev, *Perturbative aspects of the Chern-Simons field theory*, *Phys. Lett.* **B 227** (1989) 111.

JHEP07(2000)046

[9] M. Hayakawa, *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R^4*, hep-th/9912167.

[10] A. Matusis, L. Susskind and N. Toumbas, *The IR/UV connection in the non-commutative gauge theories*, hep-th/0002075.

[11] S. Minwalla, M.V. Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, hep-th/9912072.

[12] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products.* Academic Press, New York 1965.

[13] N.N. Lebedev, *Special functions and their applications*, Dover Publications, New York 1972.

[14] C.-S. Chu, *Induced Chern-Simons and WZW action in noncommutative spacetime*, *Nucl. Phys.* **B 580** (2000) 352 [hep-th/0003007].

JHEP07(2000)046